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Separation and synchronization of chaotic signals by optimization

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In this paper synchronization of multiplexed chaotic systems with smooth nonlinearities is studied. The strategy to establish if such synchronization is achievable is based on the master stability function approach and on the optimization of the coupling parameters. With this approach we are able to show that systems formed by three independent canonical chaotic circuits (i.e., a Lorenz system, a Rössler oscillator, and a Chua's circuit) can be synchronized through a unique scalar signal.

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I. INTRODUCTION

In this paper synchronization [1,2] of dynamical systems formed by n distinct chaotic circuits through the transmission of a unique scalar signal is dealt with. This problem, referred to as separation and synchronization or as synchronization of multiplexed chaotic signals, has been studied in [3–5].

Multiplexed systems are coupled through the feedback of an error signal built as the difference of the corresponding state variables as in the negative feedback scheme [6]. In general, the synchronization of two groups of such chaotic systems with a negative feedback scheme requires n independent feedback signals. The problem of separation and synchronization refers to the possibility of using a scalar signal to synchronize the multiplexed chaotic systems.

In particular, Tsimring and Sushchik [3] investigated the case of simultaneous synchronization of chaotic maps. Carroll and Pecora [4] discussed how to combine two chaotic systems with the multiplexing technique to make a communication system. Arena *et al.* [5] discussed the case of piecewise linear (PWL) chaotic systems proposing an approach based on the simultaneous stabilization of many linear asymptotic observers. The problem of synchronizing multiplexed PWL systems is formulated in terms of a set of linear matrix inequalities (LMIs). If the set of LMIs admits a feasible solution, the separation and synchronization problem for the considered multiplexed systems has a solution [5]. The method provides the values of the coupling parameters, but holds only for master-slave synchronization.

In this paper we study separation and synchronization for dynamical systems with smooth nonlinearities. To this aim a totally different approach is introduced. In fact, the approach described in [5] holds only for piecewise linear systems and cannot be directly applied to dynamical systems with smooth nonlinearity.

The approach introduced in this work aims to establish if and under which conditions synchronization can be achieved by using a unique feedback signal (i.e., for instance a linear combination of the state variables) and is based on the investigation of the synchronization properties through the analysis of the master stability function (MSF) [7]. The synchronization properties of multiplexed systems depend on the choice of the variables used to build the scalar signal. Given a fixed choice of these variables, the analysis based on the MSF allows us to investigate if synchronization can be achieved. When such synchronization cannot be achieved, the choice of the parameters (state variables and weights of the linear combination) can be optimized. The strategy proposed in this paper is based on evolutionary optimization algorithms with a fitness function based on the MSF of the considered system.

Some examples for the case of $n=2$ are shown. For the case of $n=3$ a numerical simulation showing the separation of the three chaotic dynamics and the synchronization of the three pairs of chaotic systems is also shown.

The approach proposed in this paper can be adopted also in the case of multiplexed systems coupled through a network of connections, unlike the approach in [5] which can be applied only to the master-slave configuration. A numerical example of separation and synchronization in a ring of multiplexed chaotic systems is also reported.

The paper is organized as follows. In Sec. II the approach based on the MSF analysis is introduced. In Sec. III the analysis of synchronization properties of several multiplexed systems is presented. In Sec. IV some examples of separation and synchronization of multiplexed chaotic systems made of canonical chaotic circuits are given. In Sec. V synchronization in a network of multiplexed chaotic system is studied. In Sec. VI conclusions of the paper are drawn.

II. THE APPROACH

In this paper we investigate the synchronization properties of systems made of n different and uncoupled chaotic subunits. These systems are coupled through scalar signals obtained by summing the state variables of the chaotic subunits. The scalar signal can then be transmitted into a unique channel used to synchronize multiple independent chaotic subsystems. Since multiple signals are sent to the same channel, we refer to these systems as multiplexed systems.

We first describe how the scalar signal is used to connect two multiplexed systems. The scheme proposed is based on negative feedback. Let us consider two multiplexed systems as shown in Fig. 1, where a directional coupling from the multiplexed system A to the multiplexed system B is taken into account. The two multiplexed systems are considered identical, i.e., they contain the same n independent

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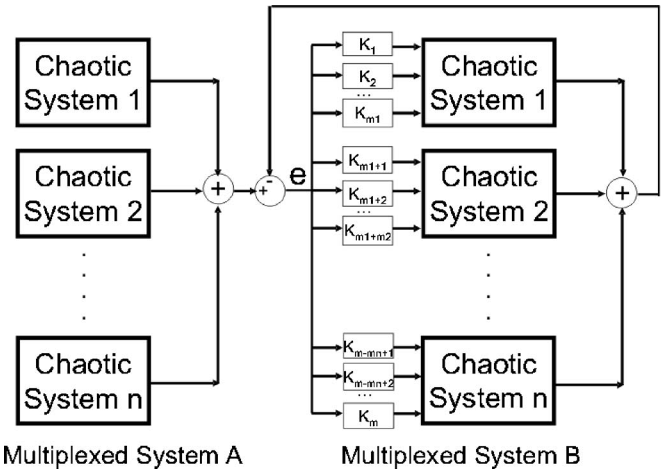

FIG. 1. Separation and synchronization of two multiplexed systems. 1-1′, 2-2′, and n-n' are identical chaotic systems starting from different initial conditions; the error is the difference between a linear combination of the state variables of system A and the same combination of the corresponding variables of system B.

subsystems with equal parameters. A linear combination of the state variables of system A is sent to system B, where an error signal is built by comparing the received signal with the same linear combination of the corresponding state variables of system B. The error signal is weighted by suitable gains and added to each state variable of system B as in the negative feedback scheme for two chaotic systems [6].

Therefore, assuming that the equations of system A are

$$\dot{\mathbf{X}}_{\mathrm{A}} = f(\mathbf{X}_{\mathrm{A}}), \tag{1}$$

system B will be described by the following equations:

$$\dot{\mathbf{X}}_{\mathrm{B}} = f(\mathbf{X}_{\mathrm{B}}) + \mathrm{K}e, \tag{2}$$

where K is the gain vector and e is the (scalar) error signal. Assuming that each multiplexed system is composed of n systems of order $m_1, m_2, \ldots, m_n$, then $\mathbf{X}_{\mathrm{A}} \in \mathbb{R}^m$ with $m = m_1 + m_2 + \cdots + m_n$, $\mathbf{X}_{\mathrm{B}} \in \mathbb{R}^m$, $\mathrm{K} \in \mathbb{R}^m$, and $e \in \mathbb{R}$.

The synchronization properties of multiplexed systems are here studied with the master stability function. The approach, introduced in [7], considers N identical oscillators coupled with the same function of the components from each oscillator to the other oscillators into an arbitrary network which admits the synchronization manifold as an invariant manifold. The approach is based on the linearization of the network dynamics around the synchronization manifold.

In [7] the dynamics of each node is modeled as $\dot{\mathbf{x}}^i = F(\mathbf{x}^i) + \sigma \Sigma_j G_{ij} H(\mathbf{x}^j)$ where $\dot{\mathbf{x}}^i = F(\mathbf{x}^i)$ represents the dynamics of each isolated node, σ is the coupling strength, $H: \mathbb{R}^m \to \mathbb{R}^m$ is the coupling function, and $G = [G_{ij}]$ is a zero-row-sum matrix modeling network connections. The synchronization properties of this network are studied by calculating the maximum Lyapunov exponent λ_{max} of the generic variational equation

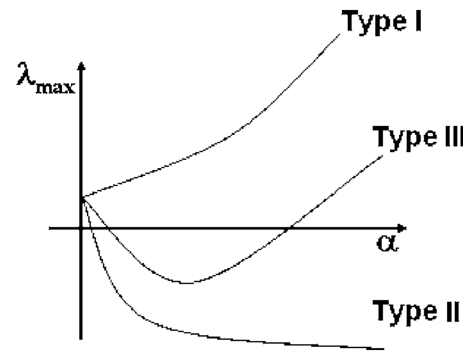

FIG. 2. Classification of oscillators with respect to the functional dependence of the maximum Lyapunov exponent λ_{max} on α.

$$\dot{\zeta} = [DF + (\alpha + i\beta) DH]\zeta \tag{3}$$

as a function of α and β, where DF and DH represent the Jacobians of $F(\mathbf{x}^i)$ and $H(\mathbf{x}^j)$ computed around the synchronous state. Once λ_{max} is obtained as a function of $\alpha + i\beta$, i.e., $\lambda_{max} = \lambda_{max}(\alpha + i\beta)$, which does not depend on the connection network, the stability of the synchronization manifold in a given network can be evaluated by computing the eigenvalues γ_h (with $h = 2, \ldots, N$) of the matrix G and studying the sign of λ_{max} at the points $\alpha + i\beta = \sigma\gamma_h$. If all eigenmodes with $h = 2, \ldots, N$ are stable, then the synchronous state is stable at the given coupling strength. In fact, we recall that, since G is a zero-row sum, the first eigenvalue is $\gamma_1 = 0$ and represents the variational equation of the synchronization manifold.

In particular, if G has real eigenvalues, as a in our case, the MSF can be computed only as a function of α. In the following we will restrict our analysis to this case. As shown in Fig. 2, the functional dependence of λ_{max} on α can give rise to three different cases [8]. The first case, denominated as type I, is the case in which network nodes cannot be synchronized. In the second case (type II) increasing the coupling coefficient σ always leads to a stable synchronous state. In the third case (type III), network nodes can be synchronized only if $\sigma\gamma_h$ for $h = 2, \ldots, N$ lie in the interval with negative values of λ_{max}.

Referring to the formulation of the synchronization problem of multiplexed systems in terms of Eqs. (2), the matrix DH becomes

$$DH = \begin{bmatrix} k_1 b_1 & k_1 b_2 & \ldots & k_1 b_m \\ k_2 b_1 & k_2 b_2 & \ldots & k_2 b_m \\ \vdots & \vdots & \ddots & \vdots \\ k_m b_1 & k_m b_2 & \ldots & k_m b_m \end{bmatrix} \tag{4}$$

where $k_1, k_2, \ldots, k_m$ are the gains $(K = [k_1\ k_2 \cdots k_m])$ and $b_1, b_2, \ldots, b_m$ with $b_i = \{0, 1\}$ specify which state variables are used to build the error signal e. Let us define $B = [b_1\ b_2 \cdots b_m]$.

The master stability equation will be in general a function of K and B. At this point, the problem of the possibility of synchronizing multiplexed systems can be translated into the problem of the existence of suitable values of K and B for which the MSF is either type II or type III.

TABLE I. Synchronization properties of multiplexed systems made of two chaotic subsystems with respect to different definitions of the error signal with $K=K_1$. The compact notation indicates the variables used, for instance $E_{xx}=E(x_i,x_j)$ where i and j represent the two subsystems used.

	E_{xx}	E_{yy}	E_{zz}	E_{xyxy}	E_{xzxz}	E_{yzyz}	E_{xyzxyz}
Lorenz-Chua	I	I	I	I	I	I	I
Chua-Rössler	I	I	I	III	I	I	I
Lorenz-Rössler	I	III	I	I	I	I	III

To solve this problem, an approach based on evolutionary optimization algorithms is used. The optimization procedure is based on the evolution of a population of individuals coding the possible solutions to the problem [9]. During the population evolution, crossover and mutation ensure the search for new solutions, while selection is performed according to a given fitness function which defines the optimization problem. In our case the problem is to obtain a MSF with $\lambda_{max}<0$. To this aim the fitness function can be defined as follows:

$$f=\min \lambda_{max}. \tag{5}$$

Given this fitness function, the optimization procedure is carried out to search for K and B that minimize the maximum Lyapunov exponent of Eq. (3). If the optimum value is such that $\lambda_{max}<0$, then the problem of synchronizing multiplexed systems will have a solution. Of course, the existence of this solution is not related to the stability of the synchronization manifold in a given complex network, but $\lambda_{max}<0$ ensures that there exists synchronizable networks with multiplexed systems as nodes.

III. ANALYSIS OF MASTER STABILITY FUNCTION OF MULTIPLEXED SYSTEMS

The proposed approach has been applied to multiplexed systems of three well-known chaotic systems: a Lorenz system [10], a Rössler oscillator [11], and a Chua's circuit [12]. The dimensionless equations describing these three systems are reported in the following with the parameters adopted in the paper. All the parameters are chosen so that chaotic behavior is obtained.

The Lorenz system is described by the following equations:

$$\dot{x}_L=\sigma(y_L-x_L),$$

$$\dot{y}_L=\rho x_L-y_L-x_L z_L,$$

$$\dot{z}_L=x_L y_L-b_L z_L, \tag{6}$$

with $\sigma=10$, $b_L=8/3$, $\rho=28$.

The following dimensionless equations model the Rössler oscillator:

$$\dot{x}_R=-(y_R+z_R),$$

$$\dot{y}_R=x_R+a_R y_R,$$

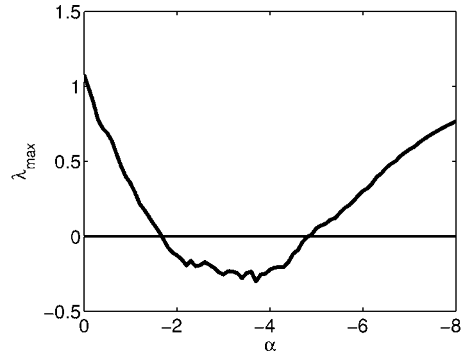

FIG. 3. Master stability function considering a multiplexed system formed by a Lorenz system and a Rössler oscillator. The error signal is defined as E_{yy} and $K=K_1$.

$$\dot{z}_R=b_R+z_R(x_R-c_R), \tag{7}$$

with $a_R=b_R=0.2$, $c_R=7$.

The Chua's circuit is described by the following equations:

$$\dot{x}_C=\alpha(y_C-a_C x_C^3-c_C x_C),$$

$$\dot{y}_C=x_C-y_C+z_C,$$

$$\dot{z}_C=-\beta y_C, \tag{8}$$

with $\alpha=10$, $\beta=16$, $a_C=1$, $c_C=-0.143$.

We applied the MSF approach to several multiplexed systems obtained by pairing two of the three chaotic systems described above. Moreover, these multiplexed systems differ from the way in which the two subsystems are coupled. In other words depending on the definition of the error scalar signal adopted, different multiplexed systems are obtained, which in general have different synchronization properties.

In the following, the notation adopted to define the error signal will be

$$e=E(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)$$

where

$$\begin{aligned}E(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)\\ =x_{1,m}-x_{1,s}+\cdots+x_{n,m}-x_{n,s}+y_{1,m}-y_{1,s}+\cdots\\ +y_{n,m}-y_{n,s}+z_{1,m}-z_{1,s}+\cdots+z_{n,m}-z_{n,s},\end{aligned} \tag{9}$$

and $x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n$ represent all the subsystem state variables effectively used to build the error signal. For instance, if we consider two subsystems (a Lorenz system and a Chua's circuit), coupled through x_L and x_C; thus $E(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)=E(x_L,x_C)$ where

$$E(x_L,x_C)=x_{L,m}-x_{L,s}+x_{C,m}-x_{C,s}. \tag{10}$$

The first case analyzed was the case in which we fixed $K=K_1=[1\ 1\ 1\ 1\ 1\ 1]$. Based on the results obtained, the multiplexed systems can be classified according to their synchronization properties as shown in Table I.

As an example, we can consider the case of a multiplexed system made of a Lorenz system and a Rössler oscillator,

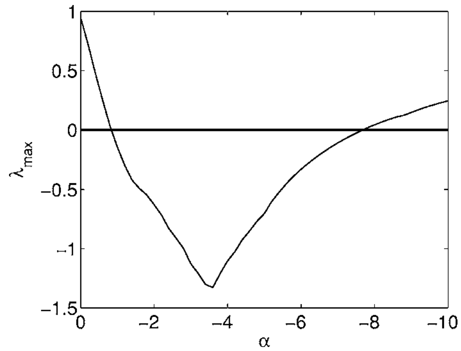

FIG. 4. Master stability function for a multiplexed system formed by a Lorenz system and Rössler oscillator. The error signal is defined as E_{xyxy} and $K=\bar{K}_{LR}$.

coupled through $E(y_L,y_R)$ which as shown in Table I is type III. The equations describing the considered multiplexed system are the following:

$$\dot{x}_L=\sigma(y_L-x_L),$$

$$\dot{y}_L=\rho x_L-y_L-x_L z_L,$$

$$\dot{z}_L=x_L y_L-b_L z_L,$$

$$\dot{x}_R=-(y_R+z_R),$$

$$\dot{y}_R=x_R+a_R y_R,$$

$$\dot{z}_R=b_R+z_R(x_R-c_R). \tag{11}$$

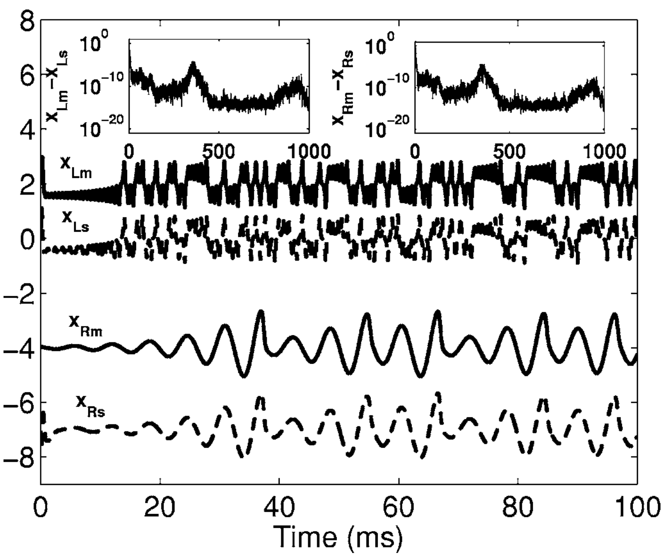

FIG. 5. Separation and synchronization of two multiplexed systems formed by a Lorenz system and a Rössler oscillator and coupled in a master-slave configuration. The feedback signal is defined as E_{yy} and the coupling strength is $\sigma=3$. Trends of state variables x_L and x_R are shown (for better visualization signals are normalized as follows: $x_{Lm}/20+2$, $x_{Ls}/20$, $x_{Rm}/10-4$, and $x_{Rs}/10-7$). Logarithmic plots of the absolute values of synchronization errors are shown in the insets.

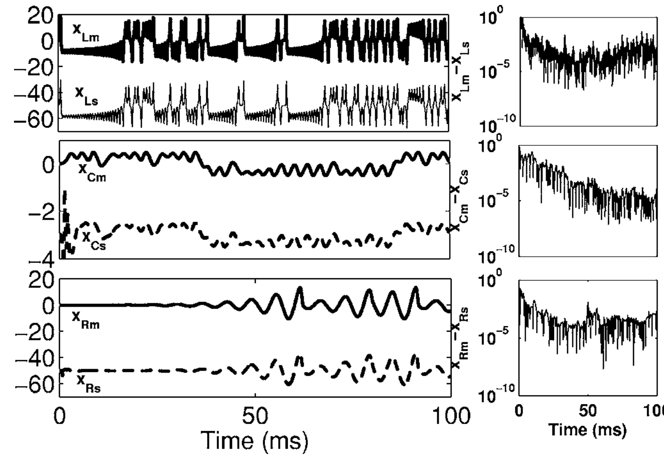

FIG. 6. Separation and synchronization of two multiplexed systems made of three chaotic subsystems with $K=\bar{K}_{LCR}$: a Lorenz system, a Chua's circuit, and a Rösslër oscillator. Trends of x_L, x_C, and x_R (and logarithmic plots of the errors, insets) are shown (for better visualization an offset has been added to slave state variables).

As reported in Sec. II, the dymanics of each node is modeled as $\dot{\mathbf{x}}^i=F(\mathbf{x}^i)$. In this case $\mathbf{x}^i=[x_L\ y_L\ z_L\ x_R\ y_R\ z_R]^T$.

The choice of $E(y_L,y_R)$ as the error signal means that the vector B defined in the approach description becomes $B=[0\ 1\ 0\ 0\ 1\ 0]$. The gains vector K is chosen equal to K_1. Thus the matrix DH is

$$DH=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

Considering only matrices G with real eigenvalues, the MSF behavior for such multiplexed system is reported in Fig. 3. As can be noticed, there exists a range of α corresponding to negative values of λ_{max}. This means that synchronization can be achieved choosing the coupling strength σ so that all the eigenvalues $\sigma\gamma_h$ of the matrix G are inside this range.

As can be noticed in Table I, in most cases the multiplexed system is type I and therefore synchronization cannot be achieved. This can be due to the choice of the coupling parameters (i.e., K). For this reason, when the multiplexed system is type I, we investigated the possibility of finding a coupling vector such that the multiplexed system is type II or III, by applying the optimization strategy with the fitness function defined as in (5). With this approach suitable coupling parameters are found in several cases.

For instance, in the case of a multiplexed system made of a Lorenz system and a Rössler oscillator, coupled through $E(x_L,y_L,x_R,y_R)$ (which is type I for $K=K_1$), we were able to find a suitable coupling vector $K=\bar{K}_{LR}$ such that the multiplexed system is type III. The evolutionary optimization algorithm was used with crossover probability $p_c=0.7$, mutation probability $p_m=0.7$, and generation gap $g_{gap}=0.9$. The elements of vector K were searched in the range $[-2;2]$ generating 20 individuals that evolve for 30 generations. Chromosomes are represented with 16-bit precision, so that it is possible to select 2^{16} values inside the fixed range.

The algorithm minimized the fitness function (5) finding the vector $\bar{K}_{LR}=[1.4346\ 1.2546 - 0.1287\ 0.9619\ 0.1043\ -0.0445]$. The resulting MSF is shown in Fig. 4 and, as can be noticed, is type III. This means that using $K=\bar{K}_{LR}$ the considered multiplexed system can be synchronized setting the eigenvalues $\sigma\gamma_h$ of connection matrix G inside the range of α corresponding to the negative values of the MSF.

Using the optimization procedure, we were also able to obtain a set of gains K solving the separation and synchronization problem for a multiplexed system with $n=3$. We considered a multiplexed system made of a Lorenz system, a Chua's circuit, and a Rössler oscillator, coupled through $E(x_L, y_L, x_C, y_C, x_R, y_R)$ (which is type I for $K_i=1$ $\forall i=1,\ldots,9$). The optimization procedure gives a value of

$$\bar{K}_{LCR}=[-0.47 \quad 1.72 \quad 0.24 \quad 1.57 \quad 0.18 \quad 0.07 \quad 1.16 \quad 0.57 \quad -0.15]$$

which leads to a MSF of type III.

IV. MASTER-SLAVE SYNCHRONIZATION

When two multiplexed systems are coupled in a master-slave configuration, the matrix G that defines the connections between the systems is

$$G=\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

which has $\gamma_1=0$ and $\gamma_2=-1$. Thus the eigenvalue of matrix G that has to be set inside the synchronization range, considering the coupling strength σ, is $\sigma\gamma_2=-\sigma$.

Referring to the examples described in Sec. III, we tested the synchronization capabilities of two multiplexed systems connected in a master-slave configuration. In the first of the examples discussed above the multiplexed system is formed by a Lorenz system and a Rössler oscillator coupled using $E(y_L, y_R)$ as the error signal. The equations describing the master are those reported in (11) while the slave is characterized by the following equations:

$$\begin{aligned}
\dot{x}_L &= \sigma(y_L - x_L) + k_1 E(y_L, y_R),\\
\dot{y}_L &= \rho x_L - y_L - x_L z_L + k_2 E(y_L, y_R),\\
\dot{z}_L &= x_L y_L - b_L z_L + k_3 E(y_L, y_R),\\
\dot{x}_R &= -(y_R + z_R) + k_4 E(y_L, y_R),\\
\dot{y}_R &= x_R + a_R y_R + k_5 E(y_L, y_R),\\
\dot{z}_R &= b_R + z_R(x_R - c_R) + k_6 E(y_L, y_R).
\end{aligned} \tag{13}$$

Thus, in the general equation $\dot{\mathbf{x}}^i = F(\mathbf{x}^i) + Ke$, with $\mathbf{x}^i = [x_L\ y_L\ z_L\ x_R\ y_R\ z_R]^T$.

The coupling strength was chosen as $\sigma=3$, according to Fig. 3. In fact, the eigenvalue $\sigma\gamma_2=-\sigma=-3$ lies inside the range corresponding to negative values of the MSF. Synchronization of the two systems is shown in Fig. 5. As can be noticed, the two circuits having different time scales can be perfectly synchronized.

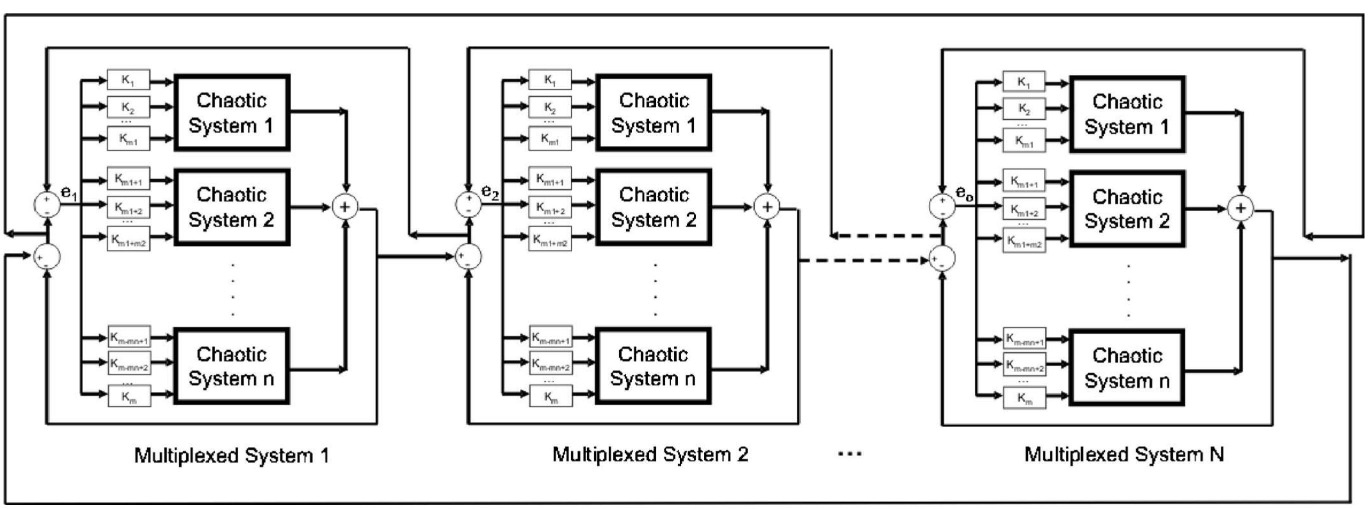

FIG. 7. Separation and synchronization scheme for a ring of N multiplexed systems.

As a second example the case with $n=3$ is discussed. A suitable value of the coupling strength to achieve separation and synchronization with $K=K_{LCR}$ is $\sigma=2.03$. Numerical results confirm that this choice leads to complete synchronization of the three pairs of chaotic systems. Figure 6 shows the state variables x_L, x_C, and x_R of the master which are perfectly synchronized with the corresponding slave variables. Even in this case, the circuits do not need to have the same time scale.

V. SEPARATION AND SYNCHRONIZATION IN A RING OF COUPLED MULTIPLEXED SYSTEMS

Multiplexed systems can also be considered as the nodes of an arbitrary complex network: synchronization can be achieved if the matrix G that describes the network topology has all the eigenvalues placed into the range of negative values of the MSF.

In this section we report the results obtained connecting several multiplexed systems in a ring topology as shown in Fig. 7. The synchronization properties of many complex networks are studied in many works in literature [2,8] which, however, consider the case of single chaotic nodes.

In the case of a ring topology the matrix G becomes

$$G=\begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 0 & 1 & -2 \end{pmatrix}$$

whose eigenvalues [13] are $\gamma_i=-4\sin^2(\pi i/N)\ \forall i=0,\ldots,N-1$ where N is the order of the matrix G corresponding to the number of coupled oscillators in the ring.

In order to calculate the maximum number of oscillators N_{max} that can be synchronized in a ring topology, let us consider type III systems. Defining α_1 and α_2 as the upper and lower bound of the range of α correspondent to negative values of the MSF, we have to ensure that $\alpha_1>\sigma\gamma_i>\alpha_2$ $\forall i=1,\ldots,N-1$. In particular, considering γ_{max} and γ_{min} as the maximum and the minimum eigenvalue of matrix G, the condition becomes $\gamma_{max}/\gamma_{min}<\alpha_2/\alpha_1$.

In the case of ring connections we have

$$\gamma_{min}=\gamma_1=-4\sin^2(\pi/N)$$

and

$$\gamma_{max}=\gamma_{N/2}=-4\sin^2(\pi N/2N)=-4.$$

Hence substituting these values in the previous condition and evaluating N we obtain $N<\pi/\arcsin\sqrt{\alpha_1/\alpha_2}$.

Thus the maximum number of oscillators that can be coupled in a ring network obtaining the synchronization of all of them is given by

$$N_{max}=\left\lfloor\frac{\pi}{\arcsin\sqrt{\alpha_1/\alpha_2}}\right\rfloor. \tag{14}$$

We report the results obtained considering the case of a ring of multiplexed systems made of a Lorenz system and a Rössler oscillator and coupled using an error signal defined as $E(x_L,y_L,x_R,y_R)$ with $K=\bar{K}_{LR}$ calculated using the optimization procedure described in Sec. III. The MSF, reported in Fig. 4, shows a suitable range for α between $\alpha_1\simeq-0.9$ and $\alpha_2\simeq-7.6$. Applying Eq. (14) we obtain $N_{max}=8$.

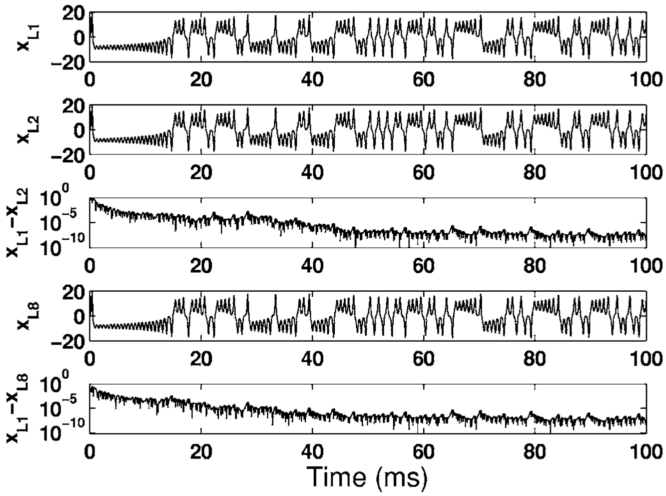

FIG. 8. Synchronization of a ring of eight multiplexed systems formed by a Lorenz system and a Rössler oscillator coupled through an error signal defined as E_{xyxy} and with coupling strength $\sigma=4$ and $K=\bar{K}_{LR}$. Trends of state variables x_{L1}, x_{L2}, and x_{L8} and logarithmic plots of the absolute errors $|x_{L1}-x_{L2}|$ and $|x_{L1}-x_{L8}|$.

Results are shown in Fig. 8, where the trends of state variables x_{L1}, x_{L2}, and x_{L8} and logarithmic plots of the absolute errors $|x_{L1}-x_{L2}|$ and $|x_{L1}-x_{L8}|$ are reported.

VI. CONCLUSIONS

In this paper, separation and synchronization of chaotic systems with smooth nonlinearities is investigated. An approach based on the optimization of the MSF of the multiplexed chaotic system is shown to be suitable to establish when multiplexed systems can be synchronized.

With respect to the technique presented in [5] (suitable for piecewise linear chaotic systems), the introduced approach can be applied to dynamical systems with smooth nonlinearities and gives conditions for the synchronization not only in the case of master-slave coupling, but also in the more general case of networks made by multiplexed systems. On the other hand, while the technique introduced in [5] gives the values of the coupling strengths, in this case an optimization strategy should be used.

With the introduced approach we are able to show that multiplexed systems formed by chaotic circuits which are the canonical chaotic systems (i.e., the Lorenz system, the Rössler oscillator, and the Chua's circuit) can be effectively synchronized. This is to our knowledge the first example of synchronization of multiplexed systems with $n=3$ units.

Furthermore, as shown in [5], separation and synchronization can be adopted in chaotic communication systems to transmit two or more different information in two or more different chaotic signals.

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